A "Superfat" Attractor with a Singular-Continuous 2-D Weierstrass Function in a Cross Section

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A 3-D invertible map generating a 4-loop solenoid attractor is presented. The attractor of the "standard" 4-solenoid turns out to be "fat" – that is, fractal dimensionality exceeds topological dimensionality by more than unity. If a fourth variable that is weakly damped is added to this 3-variable map, a new type of attractor is generated. In a cross section it is a continuous nowhere-differentiable function over a 2-D domain which is a 2-D Cantor set. Such an object can be called a "singular-continuous 2-D Weierstrass function". The latter can be "superfat" (dimension gap larger than 2). Realistic systems possessing attractors of this new type are possible.

1. Introduction

A "fat" attractor is an attractor whose fractal dimensionality exceeds its topological dimensionality by more than unity. An example was proposed recently [1]. It consists of Hénon's [2] 2-variable map, augmented by a weakly damped third variable that is passively forced. According to the Kaplan-Yorke conjecture [3], an increase in attractor dimensionality beyond the next integer is expected to occur under this condition since the negative eigenvalue of the equilibrium of the passively forced variable is smaller in absolute value than the possitive Lyapunov characteristic exponent of the forcing chaos. This is indeed the case. Nevertheless topological dimensionality remains unchanged [1]. Therefore the attractor is fat.

In the same vein, a "superfat" attractor can be generated: It suffices to add *two* weakly damped variables to Hénon's map to get a "dimension gap" between fractal and topological dimensionality of more than 2 [4]. This hierarchy can be continued.

In the following, a new type of superfat attractor will be presented which is based on a partially *different* mechanism.

2. Smale Solenoids

The simplest chaos-generating diffeomorphism, after the 2-D Smale horseshoe map [5] and its related Hénon map [2], is the Smale solenoid [5]. In this 3-D map, the zeroth iterate is a solid ring in 3-space, and the first iterate is the same ring once more, but first elongated to twice its former length and then put back into the original ring after "squeezing" (volume contraction) and one "wrapping up". The solenoid forms a prototype (everywhere expanding "axiom A") strange attractor [5]. A first explicit equation (in mixed polar and Euclidean coordinates) was given by Smale [6]. A differential equation in 4 variables generating a related attractor exists [7].

The Smale solenoid, with *one* wrapping up, is only the first in a series. The other members involve more wrapping ups. The next (in a sense) is the "4-solenoid". It involves two (consecutive) wrapping-ups. It is nontrivial because its time inverse produces hyperchaos (that is, possesses *two* positive Lyapunov characteristic exponents) [8]. It therefore generates the simplest chaotic attractor with one positive but two "nontrivial" negative exponents.

The following explicit 3-D diffeomorphism generates a Smale-solenoid when the parameter N is put equal to 2, and a 4-solenoid when $N=4$ (the case of interest here), and an N-solenoid (with ring-shaped configuration in a cross section) in general:

$$\begin{aligned} x_{n+1} &= N\, x_n \bmod 2\pi\,, \\ y_{n+1} &= K\, y_n - (1-K) \cos N x_n\,, \\ z_{n+1} &= K\, z_n + (1-K) \sin N x_n\,. \end{aligned} \qquad (1)$$

The equation has a simple structure. The first variable is the familiar chaos-generating continuous piece-wise linear map (Bernouilli shift). The second and the third variable are linear contractions (with K less than

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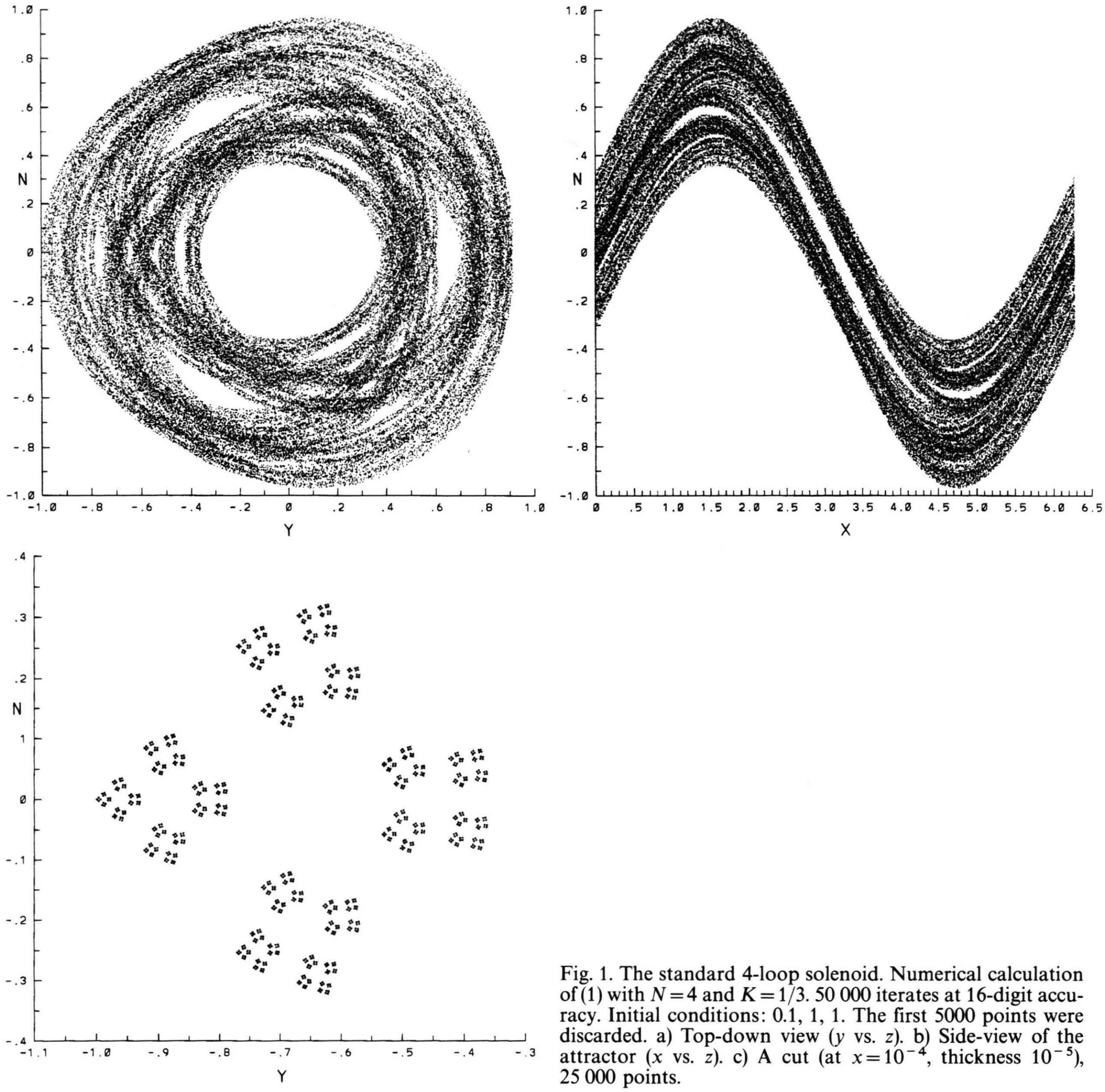

Fig. 1. The standard 4-loop solenoid. Numerical calculation of (1) with $N=4$ and $K=1/3$. 50 000 iterates at 16-digit accuracy. Initial conditions: 0.1, 1, 1. The first 5000 points were discarded. a) Top-down view (y vs. z). b) Side-view of the attractor (x vs. z). c) A cut (at $x=10^{-4}$, thickness 10^{-5}), 25 000 points.

unity) that are passively forced by x. The two trigonometric functions of x on the right make sure that the momentary value of x displaces the location of the fixed point of the 2-variable linear contraction away from the origin of the y, z plane, in such a way that it circles the origin N times as x varies from zero to 2π (cf. [3]).

The shrinking factor (K) must always be smaller than 1/2 to ensure invertibility if $N = 2$. Invertibility is lost beyond this limit due to the formation of an "overlap" between the two substrands formed. Analogous (smaller) limits apply for larger values of N. We choose the values $N=4$ (4-solenoid) and $K=1/3$, to obtain the "standard" 4-solenoid as it may be called. It is related to the "standard" (middle thirds removed) Cantor set, since the contraction operation has an effect like this in two directions (as we shall see immediately).

Figure 1 shows the resulting attractor in 3-space. Figure 1a gives a top-down view along the x-axis.

Figure 1 b shows the corresponding side-view. Note that in both pictures, the circle of x (in Smale's description) has been "stretched flat" (in accordance with the modulo mechanism of x in (1)). Finally, Fig. 1 c shows a cross section (at x close to zero).

3. Fractal Dimensionality of the Attractor of Figure 1

The standard 4-solenoid attractor of Fig. 1 turns out to be "fat". It is, topologically speaking, a product of an interval and two standard Cantor sets. The rotations visible in the cross section (Fig. 1 c) do not change this conclusion. Accordingly, the fractal dimensionality of the attractor of Fig. 1 can be calculated:

$$D_F = \frac{\log 4}{\log 3} + 1 = 2.261859\ldots, \qquad (2)$$

whereby $\log 4/\log 3 = 1.261859\ldots$ is the fractal dimensionality of the standard 2-D Cantor set ("iterated Swiss flag").

The fact that D_F is not only larger than unity but larger than 2 is surprising since the topological dimensionality of the solenoid attractor is unity [5]. The reason for the present fatness is that the standard 2-dimensional Cantor set (iterated Swiss flag) is already fat itself. It possesses a fractal dimensionality in excess of unity (namely, $\log 4/\log 3$ as mentioned). This fatness is a straightforward implication of the Hausdorff rule for calculating fractal dimensionality, cf. [9]: Each subsequent iterate of the (red) square contains more copies of the original set (four) than are needed to fill a 1-D row of copies (three). Hence fractal dimensionality cannot be smaller than unity [18].

The full attractor simply inherits the fatness of the standard 2-D Cantor set through product formation (added unit integer in (2)). Note that the attractor of Fig. 1 is *smooth* in spite of its fatness.

4. Adding a Weakly Damped Fourth Variable

The following 4-variable diffeomorphism differs from (1) by the addition of a passively forced fourth variable:

$$\begin{aligned} x_{n+1} &= 4x_n \bmod 2\pi, \\ y_{n+1} &= 1/3\, y_n - 2/3 \cos 4x_n, \\ z_{n+1} &= 1/3\, z_n + 2/3 \sin 4x_n, \\ w_{n+1} &= 0.99\, w_n + x_n. \end{aligned} \qquad (3)$$

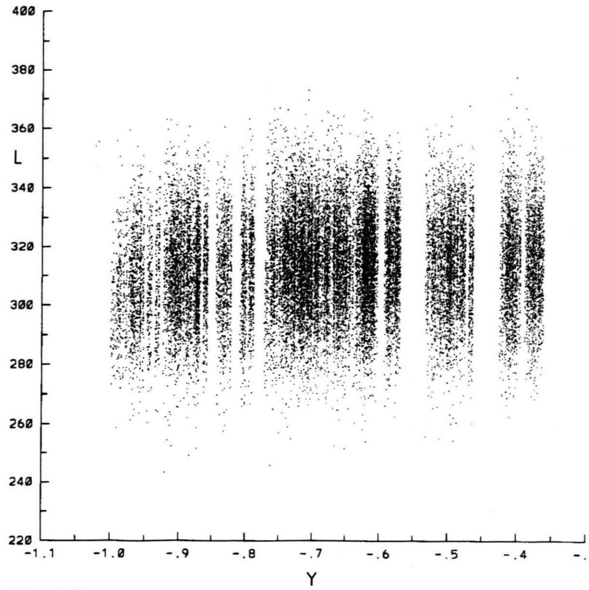

Fig. 2. The new attractor. Numerical calculation of (3). (The ordinate should read W instead of L.) Side-view of a 3-D cut (at $x = 10^{-4}$, with thickness $2 \cdot 10^{-5}$), 25 000 iterates. Initial conditions: 0.1, 1, 1, 300. The first 5000 points were discarded. The present side view (y vs. w) turns into the top-down view of Fig. 1 c (y vs. z) if w is made point out of the plane of the paper (rotation by 90 degrees, cf. text).

The weak damping of the added fourth variable (0.99 is close to unity) has the consequence that the dimensionality must go up once more in accordance with the Kaplan-Yorke conjecture [3].

Figure 2 gives a 3-D cross section (x constant close to zero) in a 2-dimensional projection (side view). One sees a bundle of "vertical needles". To better understand Fig. 2, it is helpful to realize that it represents an extremal perspective (perfect side view). If the content of the Figure were slightly tilted forward toward the viewer, one would see that the "needles" are distributed in such a way that each is suspended in 3-space vertically above one of the black pixels of Figure 1 c. The "side view" of Fig. 2 and the "top-down view" of Fig. 1 c both describe the same cross section. (The fact that Fig. 1 c was obtained from (1) rather than (3) makes no difference since w would be invisible in the present projection due to the w-axis pointing vertically out of the plane of the paper.) Both projections, taken together, numerically demonstrate the existence of a bundle of parallel "Cantor needles" in 3-space.

As a consequence of this geometric situation, one expects the fractal dimensionality of the full attractor to exceed that of (2) by unity:

$$D_F = 3.261859\ldots . \tag{4}$$

This prediction can be checked numerically. Specifically, one finds

$$D_F = 3.10 . \tag{5}$$

This result was obtained by using the nearest-neighbor algorithm [10]. The prediction (4), based on the Cantor needles hypothesis, is confirmed qualitatively – the dimension is indeed greater than 3.

5. Topological Dimensionality of the Attractor

As far as the first three lines of (3) are concerned, the topological dimensionality of the solenoid attractor is unity [5], as we saw. The full attractor generated by (3) *still* has the dimensionality

$$D_{top} = 1 . \tag{6}$$

This result follows from the theorem that a linear variable that is passively forced by a bounded forcing function of arbitrarily high periodicity $f(t)$ possesses a unique asymptotic solution (cf. [11]). Since each point on the solenoid attractor (a point in x, y, z space) generates its own unique forcing function [5], it gives rise to a unique asymptotic value for w. Hence (6) follows (cf. also [11 a]). As a consequence, each needle in Fig. 2 contains only a single point. That is, the "needles" are but *apparent* needles (one-dimensional sets).

The contradiction can be resolved if one realizes that always many needles are "bundled together" in each visible slot (above one pixel). Note that every point in a Cantor set contains in an arbitrarily small finite neighborhood an uncountable number of other points of the Cantor set. Therefore, if each point in the Cantor set gives rise to a different height value (w value), an apparent "needle" can indeed form. This result holds true under the presupposition that the corresponding height values are distributed according to a *dis*continuous function (uncorrelated neighboring height values). Unexpectedly, it *still* holds true when the pertinent function is continuous but nowhere differentiable, cf. [1].

There is one difference between the two cases, however. In the second – continuous but nowhere differentiable – case, the length of the apparent needles is no longer independent of horizontal magnification, but rather decreases toward zero according to a scaling law as horizontal magnification is increased without bounds (R. Wais, personal communication). This test has yet to be made quantitative.

6. Fractal Dimensionality of the Attractor, Revisited

"Fat fractals" [9] can arise in dynamical systems, not only in parameter space [12] but also in state space. This fact follows from the above example of a standard 2-D Cantor set occurring as a cross section in the attractor of a generic dynamical system; see (2) above. Those attractors are smooth. As mentioned in the Introduction, there exists a second class of examples. These are non-smooth since they are based on nowhere-differentiability [1].

More specifically, these latter attractors are characterized by "singular-continuous" nowhere-differentiability [1]. This new property is easy to understand. It is possible to introduce "Cantor gaps" into a nowhere differentiable function (Weierstrass function) in such a way that all maxima and minima of the latter are preserved as far as their original height values are concerned. However, they are each "split up" such as to accomodate a Cantor gap in between. An explicit construction is given in [1]. The result is a new Weierstrass function whose domain (on which it is a continuous function) is a Cantor set [1]. Its graph is a set of discontinuous points ("singular-continuous Weierstrass function"). The topological dimensionality immediately drops down to zero, even as an infinitesimal gap size is introduced, whereas the fractal dimensionality stays unchanged at first, in order to gradually decrease as gap size is increased. In the case of an ordinary Weierstrass function (1-D domain and range), the decrease is never strong enough to lead to a value below unity, irrespectively of gap size [1].

Given the existence of these *two* principles to generate fatness (one smooth, one nonsmooth), it is tempting to guess that the attractor of (3) may obey both. This conjecture turns out to be correct.

7. Structure of the New Attractor

As already mentioned, a picture of a singular-continuous Weierstrass function looks like a Cantor bundle of parallel needles, in 2-space. A product of *two* singular-continuous 1-D Weierstrass functions (a "2-D" singular-continuous Weierstrass function) looks like a Cantor bundle of parallel needles – in 3-space. This structure, taken as a "prediction", exactly matches the structure of the present attractor as found in Figs. 2 and 1 c.

This fact could still be a coincidence. To establish the connection, it would be necessary to show that there exists a "limiting case" to (3) for which the present needles "grow together" laterally to form a continuous surface that is nowhere differentiable – a Weierstrass function over a continuous 2-D domain.

Such a limiting case indeed exists. It occurs when the forcing solenoid (first three variables of (3)) becomes "dissipation-free". This occurs when the contraction factor K in (1) is put equal to $1/2$. We suppose here for simplicity that the equation behaves like the *ideal* 4-solenoid [8] up to arbitrarily low dissipation values. (It may turn out that a more complicated version of (3) is needed to cover this limiting case explicitly but this makes no difference in the present existence context.)

The forcing chaos becomes volume-filling ("Hamiltonian") in the present limit. A "Hamiltonian" forcing situation was first studied by Kaplan et al. [13]. The authors employed ordinary chaos (plane-filling case) to force a weakly damped third variable. Specifically, they used Arnold's [14] cat map, but they could as well have used Hopf's [15] baker's transformation, which is also a plane-filling bijection. (The present volume-filling limiting case of the 4-solenoid *is* a straightforward generalization of Hopf's map [8]). Kaplan et al. [13] found that an attractor is formed which possesses a "nowhere differentiable" continuous cross section. Specifically, the nowhere differentiability applies along the 1-D stable manifold of the forcing chaos while smoothness applies along the unstable manifold [16].

In the present analogous case, the same result necessarily still applies along the stable manifold of the forcing chaos. However, this situation now applies twice, due to product formation, since the stable manifold now is two-dimensional [8]. Hence there are now two directions along which nowhere-differentiability applies independently (the two 1-D "eigendirections" of the 2-D stable manifold).

To picture this limiting attractor, it suffices to return to Figs. 1 c and 2 and imagine that all the visible Cantor gaps (white) are shrunk down to zero. In the limit, the 1-D unstable manifold (the chaotic thread), whose orthogonal intersections with the 2-D stable manifold generate the black pixels, becomes space-filling (so that the gaps vanish). One then has a continuous nowhere differentiable 2-D surface (2-D Weierstrass function) as an attractor in the limit, as a nontrivial generalization of the 1-D Weierstrass function found by Kaplan et al. [13]. That is, the needles have "grown together laterally" as required.

A remaining point concerns the possibility that the present Cantor gaps (at $K=1/3$) may prove "too large" already to *still* support a dimensionality in excess of 3 (as valid close to the limit). Equation (5) shows that this fear is unjustified.

8. Discussion

A simple 3-D map generating a 4-solenoid has been combined with a weakly damped fourth variable to generate a new attractor. The latter was interpreted to be a product of the graph of a Weierstrass function over a 2-D Cantor set and a 1-D interval. As such, it could be either "superfat" ot "fat". For the parameters chosen it was superfat.

It appears that attractors governed by a singular-continuous 2-D Weierstrass function represent a new class of generic objects in differentiable dynamical systems. New quantitative questions can be addressed since these attractors undergo many interesting transitions as a function of parameters (especially K). Both new scaling laws and refined tests of the Kaplan-Yorke conjecture come into sight.

Is the new class of attractors also of practical interest? Real-life systems like chemical reactors frequently generate attractors whose numerically determined dimensionality is greater than 3 (cf. [17]). Also, continuous dynamical systems generating solenoid-like attractors exist [7]. Thirdly, weakly damped additional variables an be added, both in simulations and in experiments. It therefore is a practical necessity to know what types of attractor to expect in principle.

To conclude, a simple solenoid-based attractor has been described and interpreted in terms of a Weierstrass function that is singular-continuous in two dimensions simultaneously.

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